The distribution of cycles of length $O(n)$ in the Star graph

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Star graph

Star graph

The Star graph $S_n = Cay(Sym_n, ST)$, $n \geqslant 2$, is a Cayley graph on the symmetric group Sym_n with the generating set of transpositions $ST = \{t_i \in Sym_n, 2 \leqslant i \leqslant n\}$ exchanging i'th element of the permutation with the first.

Properties

The Star graph S_n, $n \geqslant 3$,

- is bipartite;
- contains even cycles of lengths C_l, where $6 \leqslant l \leqslant n!$;
- has diameter $D = \lfloor \frac{3(n-1)}{2} \rfloor$.

Motivation

Konstantinova, M., 2014

Each of vertices of $S_n, n \geqslant 3$, belongs to $\binom{n-1}{2}$ distinct 6–**cycles** of the following canonical form:

$$C_6 = (t_k\, t_i)^3, \quad 2 \leqslant i < k \leqslant n.$$

Konstantinova, M., 2014

Each of vertices of $S_n, n \geqslant 4$, belongs to $3(n-3)(n-2)(n-1)$ distinct 8–**cycles** of the following canonical forms:

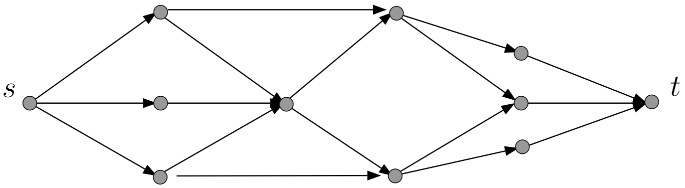
$$C_8^1 = t_k\, t_i\, t_j\, t_i\, t_k\, t_i\, t_j\, t_i, \quad 2 \leqslant i \neq j \leqslant k-1;$$
$$C_8^2 = t_k\, t_j\, t_i\, t_j\, t_k\, t_i\, t_j\, t_i, \quad 2 \leqslant i \neq j \leqslant k-1;$$
$$C_8^3 = t_k\, t_j\, t_i\, t_k\, t_j\, t_k\, t_i\, t_j, \quad 2 \leqslant i \neq j \leqslant k-1;$$
$$C_8^4 = t_k\, t_j\, t_k\, t_i\, t_k\, t_j\, t_k\, t_i, \quad 2 \leqslant i < j \leqslant k-1,$$

where $4 \leqslant k \leqslant n$.

Motivation

Oriented percolation model.

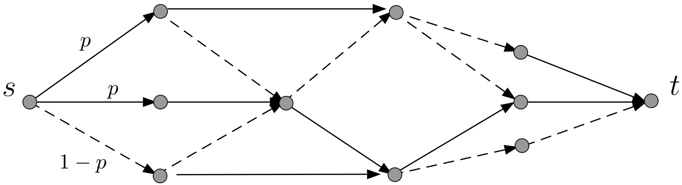
Consider a graph $G = (V, E)$ on n vertices with distinguished vertices $s, t \in V$ and edges oriented along shortest paths from s to t.



Motivation

Oriented percolation model.

Suppose every edge $e \in E$ in G is **open** with probability p, where $0 \leqslant p \leqslant 1$ and **closed** with probability $q = 1 - p$.

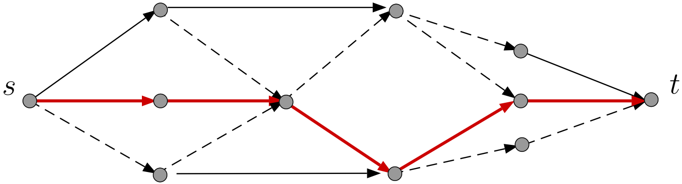


Motivation

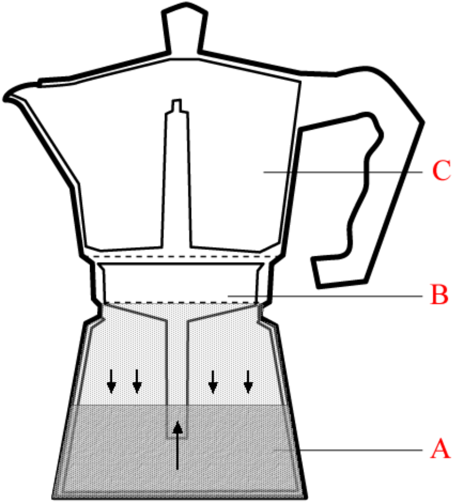
Oriented percolation model.

Question: what is the smallest value of p for which

$$\mathbf{P}_n(\exists \text{ open path from } s \text{ to } t) = 1$$



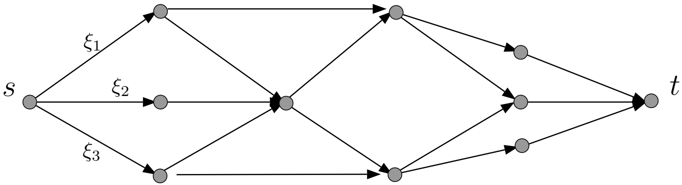
Motivation



Motivation

Oriented first passage percolation model.

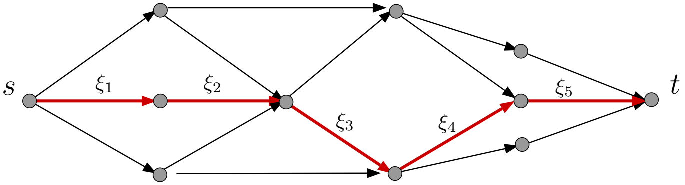
Suppose every edge $e \in E$ in G has labelled by i.i.d. random variable ξ_e, representing the passage time of the edge.



Motivation

Oriented first passage percolation model.

Question: what is the time $T = T_n$ to reach vertex t from s as $n \to \infty$?



Motivation

Oriented percolation and first passage percolation.

Fill, Pemantle, 1993

For the hypercube H_n, with $s = \bar{0}$ and $t = \bar{1}$, the critical value of p for oriented percolation is $p = \frac{e}{n}$ and for the oriented first passage percolation converges the time T_n converges:

$$T_n \xrightarrow[n \to \infty]{} 1$$

The proof is based on the distribution of $2d$-cycles in graph H_n, where $2 \leqslant d \leqslant n$.

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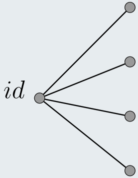
Motivation

The distribution of $2d$-cycles in S_n

Consider the graph S_n from the identity vertex id.

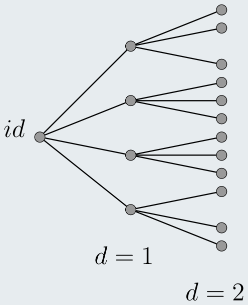


$d = 1$

Motivation

The distribution of $2d$-cycles in S_n

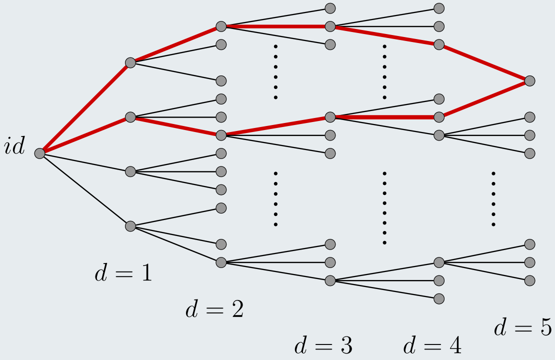
For a while the graph is locally tree-like and there is a unique shortest paths to vertices at distance d.



Motivation

The distribution of $2d$-cycles in S_n

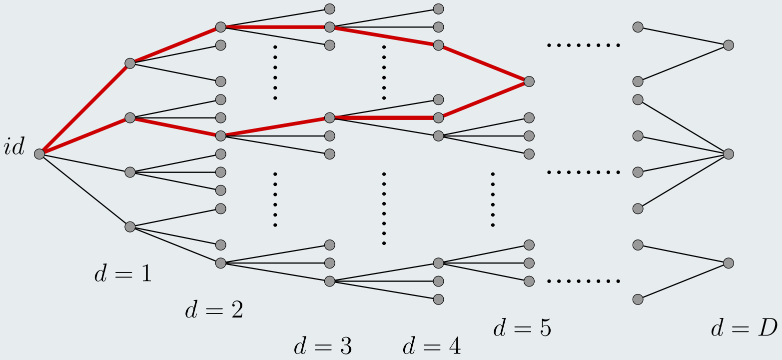
At some point shortest paths intersect at vertex creating a $2d$-cycle.



Motivation

The distribution of $2d$-cycles in S_n

Our goal is to study the distribution of such $2d$-cycles for $3 \leqslant d \leqslant D$.



Distance distribution of vertices

L.Wang, et. al., 2006

In the Star graph S_n, $n \geqslant 3$, the total number of vertices at distance d, $1 \leqslant d \leqslant D$ from identity vertex *id* is given by

$$N_d^n = \sum_{j \geqslant 0} \overset{n\,j}{\Psi}_d,$$

where

$$\overset{n\,j}{\Psi}_d = \sum_{r=2}^{\min\{d-1,n-1\}} (r-1)! \binom{n-1}{d} \times \frac{1}{j} \overset{n-r\,j-1}{\Psi}_{d-r-1}.$$

Any permutation $\pi \in Sym_n$ can be represented uniquely in terms of non-intersecting cycles, i.e.

$$\pi = (1\,\pi_1^0 \dots \pi_{\ell_0}^0)(\pi_1^1 \dots \pi_{\ell_1}^1)\dots(\pi_1^k \dots \pi_{\ell_k}^k)(.)\dots(.) =$$
$$= (1\pi^0)(\pi^1)\dots(\pi^k)$$

Denote the cycle of length ℓ containing the element "1" as $\ell - CO$ and not containing it as $\ell - CN$, then the vertices on the distance d may have either

1. only a $(d+1) - CO$;
2. an $m - CO$, $1 \leqslant m \leqslant d-2$ and $k \geqslant 1$ items of $\ell_i - CN$, where $1 \leqslant i \leqslant k$, such that $d = k + (m-1) + \sum_{i=1}^{k} \ell_i$.

Shortest Paths Algorithm

Suppose $\pi \in Sym_n$ is at distance d from the identity id. To obtain a shortest path we should apply the sequence of generating elements performing the following two operations:

1. apply the transposition $t_{\pi_1^0}$ and **contract** element π_1^0 of $\ell_0 - CO$ into its own cycle of length 1, obtaining the permutation π^*:

$$\pi^* = \pi\, t_{\pi_1^0} = (1\, \pi_2^0 \ldots \pi_{l_0}^0)(\pi_1^0)(\pi^1)(\pi^2)\ldots(\pi^k);$$

2. apply one of transpositions $t_{\pi_1^i}, \ldots, t_{\pi_{l_i}^i}$ and **merge** $\ell_0 - CO$ cycle π^0 and $\ell_i - CN$ cycle π^i, $i = 1, \ldots, k$, obtaining the permutation π^*:

$$\pi^* = \pi\, t_{\pi_j^i} = (1\, \pi_j^i\, \pi_{j+1}^i \ldots \pi_{\ell_i}^i\, \pi_1^i \ldots \pi_{j-1}^i \pi_1^0 \ldots \pi_{l_0}^0)(\pi^2)\ldots$$
$$\ldots(\pi^{i-1})(\pi^{i+1})\ldots(\pi^k),$$

where $1 \le j \le \ell_i$.

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Exact results

Denote the (π-id)-**cycle** of length $2d$ the cycle formed by two shortest paths between id and vertex π at distance d.

Theorem 1

In the Star graph S_n, $n \geqslant 3$, the number of distinct (π-id)-cycles of length $2d$, where $3 \leqslant d \leqslant n$, with vertex π having $1 - CO$ and $(d-1) - CN$ in cyclic structure is given by

$$N(1, d-1) = \frac{d-2}{2}(n-1)\dots(n-d+1).$$

Exact results

Theorem 2

In the Star graph S_n, $n \geqslant 3$, the number of distinct $(\pi\text{-}id)$-cycles of length $2d$, where $3 \leqslant d \leqslant n$, with vertex π having $(m+1)-CO$, $1 \leqslant m \leqslant d-3$ and $\ell_1 - CN$, $2 \leqslant \ell_1 = d-1-m$ in cyclic structure is given by

$$N(m, \ell_1) = \frac{d(d-3)}{2}(n-1)\ldots(n-d+1).$$

Exact results

Theorem 3

In the Star graph S_n, $n \geqslant 3$, the number of distinct (π-id)-cycles of length $2d$, where $3 \leqslant d \leqslant n+1$, with vertex π having $1-CO$, and $\ell_1 - CN$ and $\ell_2 - CN$, where $d = \ell_1 + \ell_2 + 2$, in cyclic structure is given by

$$N(1, \ell_1, \ell_2) = C_d (n-1) \ldots (n-d+2),$$

where

$$C_d = \frac{1}{24}(d-5)\left((d-2)^2 - 2\right)\left(3d^3 - 29d^2 + 51d + 114\right)$$

Asymptotic results

Theorem 4

In the Star graph S_n, $n \geqslant 3$, the number of distinct $(\pi\text{-}id)$-cycles of length $2d$, where $3 \leqslant d \leqslant n+k-1$, with vertex π having $1-CO$ and k of $\ell_i - CN$, where $d = \ell_1 + \cdots + \ell_k + k$, in cyclic structure is given by

$$N(1, \ell_1, \ell_2, \ldots, \ell_k) \asymp (k!)^2 (d-3k-2)^{4k-2} (n-1)(n-2)\ldots(n-d+k)$$

Asymptotic results

Theorem 5

In the Star graph S_n, $n \geqslant 3$, the number of distinct (π-id)-cycles of length $2d$, where $3 \leqslant d \leqslant n+k-1$, with vertex π having $m-CO$ and k of $\ell_i - CN$, where $d = \sum \ell_i + k + (m-1)$, in cyclic structure is given by

$$N(m, \ell_1, \ell_2, \ldots, \ell_k) \asymp (k!)^2 (d-3k-2)^{4k-1} (n-1) \ldots (n-d+k)$$

Thank You!

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